INSURANCE AND ACTUARIAL ENGINEERING

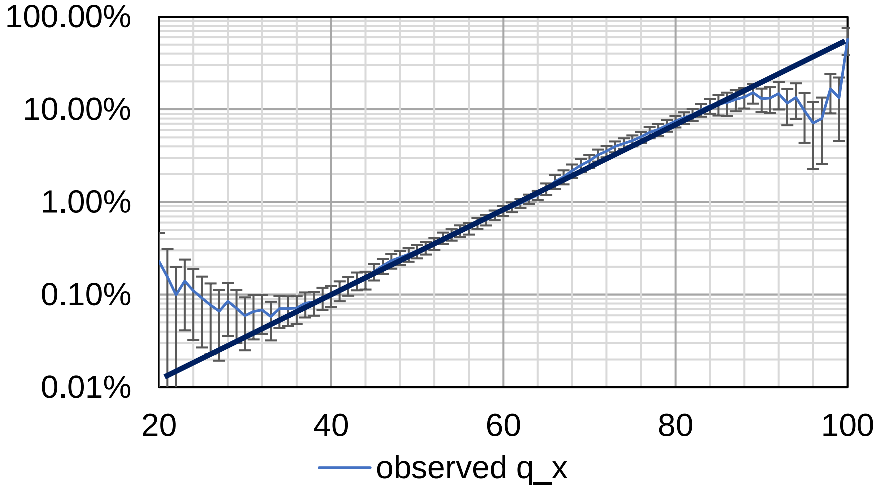
# Prime Re Solutions

# Parameter Risk

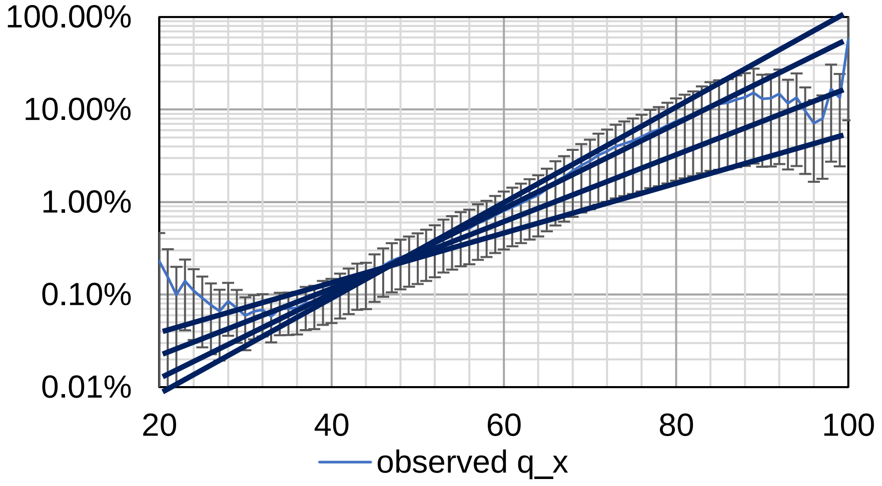
## VIII Simposio Internacional de Actuaría

# Why parameter risk

- Which portfolio would you rather handle or take over?



500’000 lives



50’000 lives

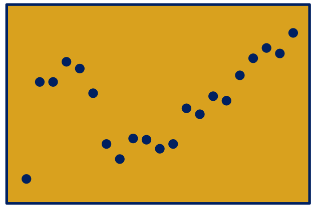
- Relevance of the parameter uncertainty
  - Pricing profit loading
  - Reserving risk margin
  - Solvency parameter risk

# Agenda

Parameters

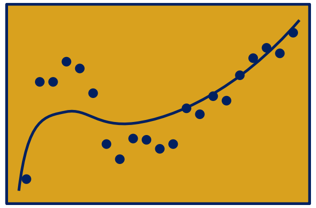
- Risk
- Parameter risk

# Chi-square fit



- $(x_i, y_i)$ are $n$ actual realizations of 2 (or more) observables

- $y = F(x, \dots)$ is some unknown function of
  - $x$
  - many more unknown variables that are difficult to observe



- $y = f_{\boldsymbol{a}}(x) + \varepsilon$ is a model that
  - focuses on the most important variable $x$
  - combines the other variables into the stochastic Gaussian process $\varepsilon \sim \mathcal{N}(0, \sigma)$
  - depends on $k$ parameters $\boldsymbol{a}$

- Calibrate the parameters $\boldsymbol{a}$ by minimizing the chi-square function $\mathrm{X}^2$ w.r.t. $\boldsymbol{a}$

$$\mathrm{X}^2 = \sum_{i=1}^{n} \left( \frac{y_i - f_{\boldsymbol{a}}(x_i)}{\underset{\text{1}\sigma\text{ observation uncertainty}}{\sigma_i}} \right)^2$$

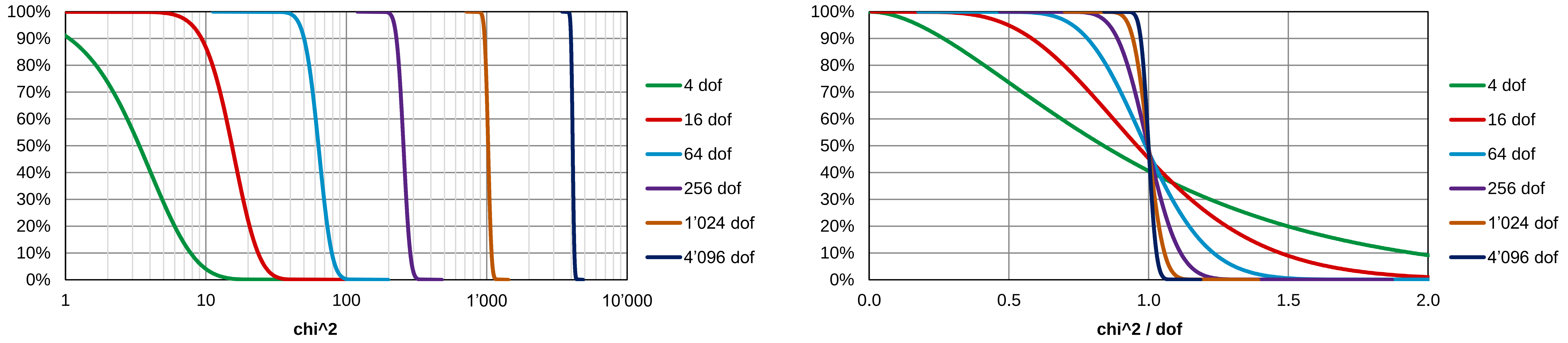
$1\sigma$ observation uncertainty

# Goodness of fit

- $\mathrm{X}^2_{min} = \min\limits_{\boldsymbol{a}}\left(\mathrm{X}^2(\boldsymbol{a})\right)$ yields the best fit parameters $\boldsymbol{a} = (a_1, a_2, \dots, a_k)$

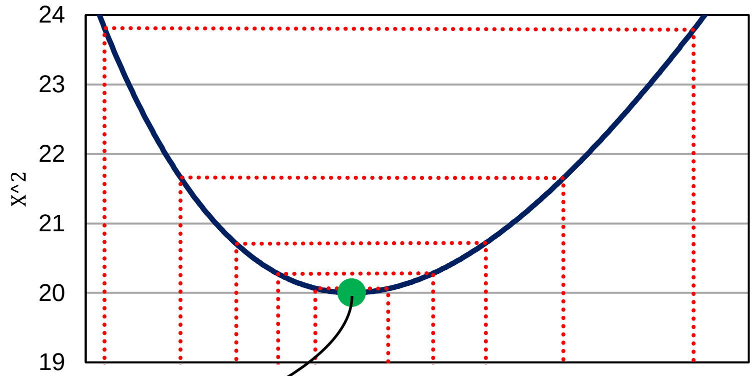
- Goodness of fit
  - $\mathrm{X}^2_{min} \sim \chi^2_{n-k}$ chi-square distribution with $n-k$ dof (cf. BIC)
  - $\mathrm{X}^2_{min}$ too large ⇒ probability too small
  - $\frac{\mathrm{X}^2_{min}}{n-k} \gg 1$ ⇒ reject the model

$n =$ number of observations
$k =$ number of parameters



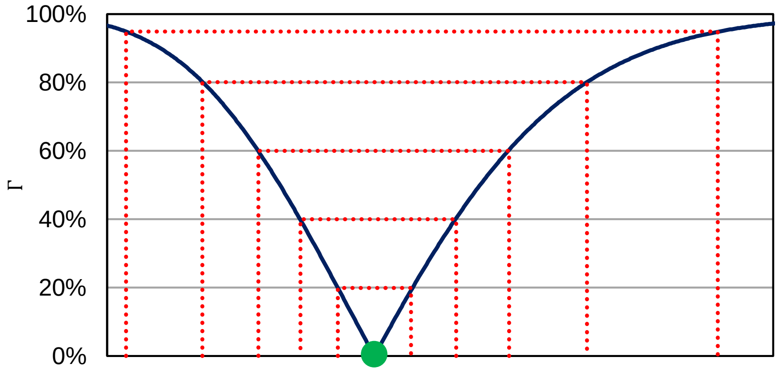
# Parameter confidence intervals

- 1 parameter
  - Confidence bounds $a_p$ at probability $p$: $\mathrm{X}^2\left(a_p\right) = \mathrm{X}_{min}^2 + \overleftarrow{\chi_1^2}(p)$
    - $\overleftarrow{\chi_1^2}(95\%) = 3.81$
    - $\overleftarrow{\chi_1^2}(80\%) = 1.65$
    - $\overleftarrow{\chi_1^2}(60\%) = 0.71$
    - $\overleftarrow{\chi_1^2}(40\%) = 0.27$
    - $\overleftarrow{\chi_1^2}(20\%) = 0.06$
    - …



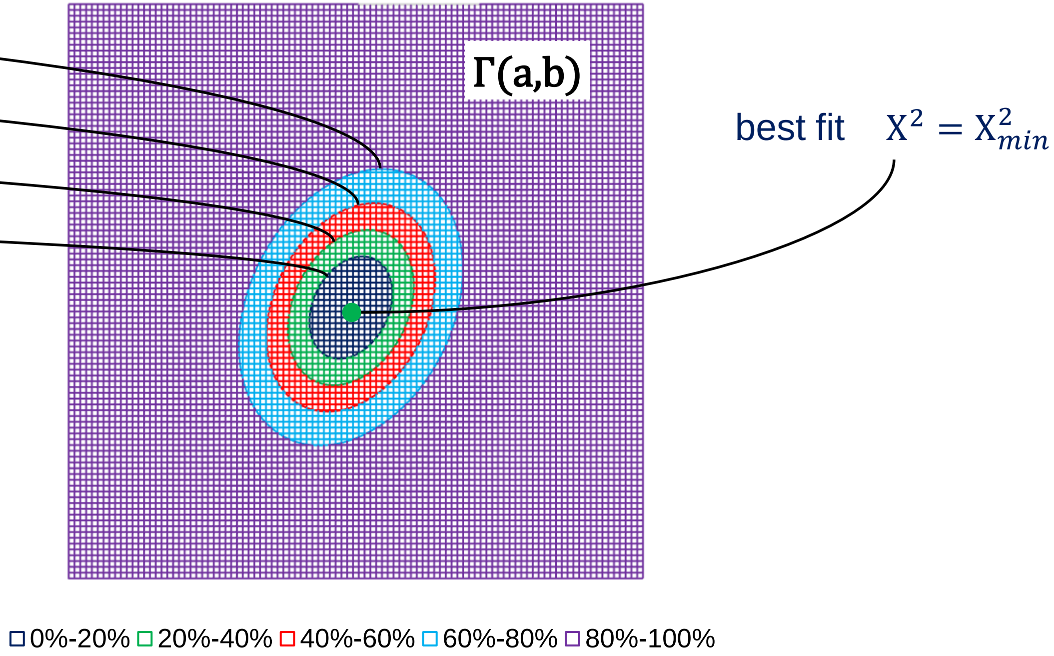
best fit $\mathrm{X}^2 = \mathrm{X}_{min}^2$

  - Confidence probability $\Gamma(a) = \chi_1^2\left(\mathrm{X}^2(a) - \mathrm{X}_{min}^2\right)$



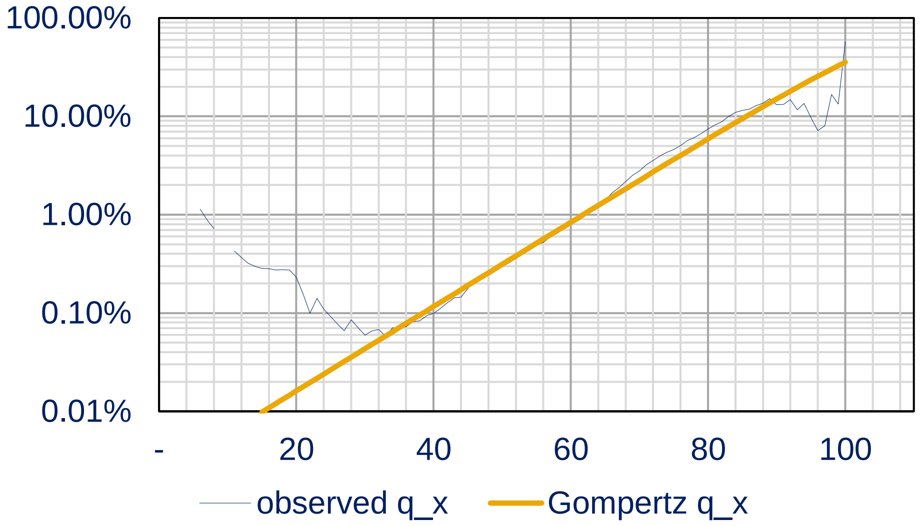
# Parameters confidence spaces

- $k$ parameters
  - Confidence contour, surface, volume, … at probability $p$: $\mathrm{X}^2\left(\boldsymbol{a}_p\right) = \mathrm{X}^2_{min} + \overleftarrow{\chi}^2_k(p)$
    - $\overleftarrow{\chi}^2_2(95\%) = 5.95$
    - $\overleftarrow{\chi}^2_2(80\%) = 3.22$
    - $\overleftarrow{\chi}^2_2(60\%) = 1.83$
    - $\overleftarrow{\chi}^2_2(40\%) = 1.02$
    - $\overleftarrow{\chi}^2_2(20\%) = 0.45$
    - …
  - Confidence probability $\Gamma(\boldsymbol{a}) = \chi^2_k\left(\mathrm{X}^2(\boldsymbol{a}) - \mathrm{X}^2_{min}\right)$



# Mortality models

- Cohort definitions
  - $L_x =$ persons of age $x$ alive
  - $q_x = \frac{L_x - L_{x+1}}{L_x} =$ probability to die between ages $x$ and $x+1$
  - $\mu_x = -\frac{1}{L_x}\frac{dL_x}{dx} =$ instantaneous mortality rate



- Gompertz assumption: $\mu_x = Me^{x/\tau}$

$\Rightarrow L_x = ce^{-M\tau e^{x/\tau}}$

$\Rightarrow q_x = 1 - e^{-M\tau(e^{1/\tau}-1)e^{x/\tau}}$

morbid death

- Makeham assumption: $\mu_x = A + Me^{x/\tau}$

$\Rightarrow L_x = ce^{-Ax}e^{-M\tau e^{x/\tau}}$

$\Rightarrow q_x = 1 - e^{-A}e^{-M\tau(e^{1/\tau}-1)e^{x/\tau}}$

accidental death

# Mortality models

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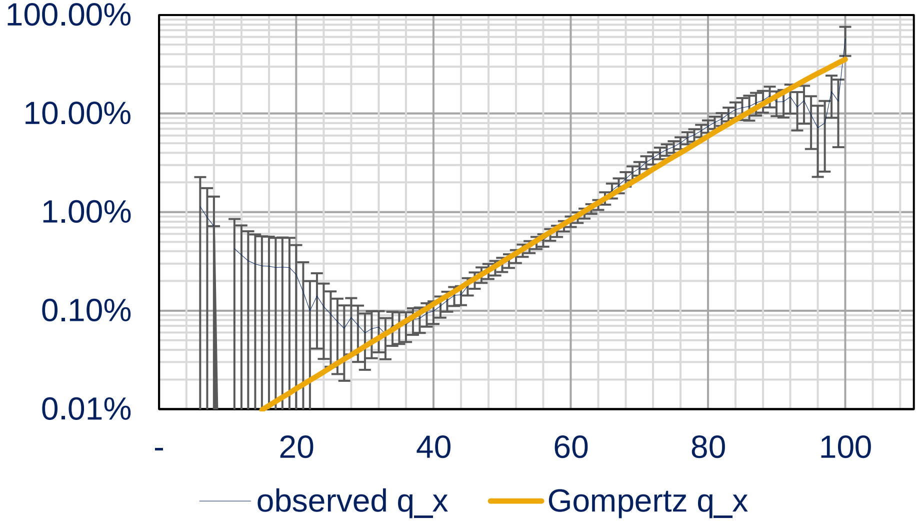
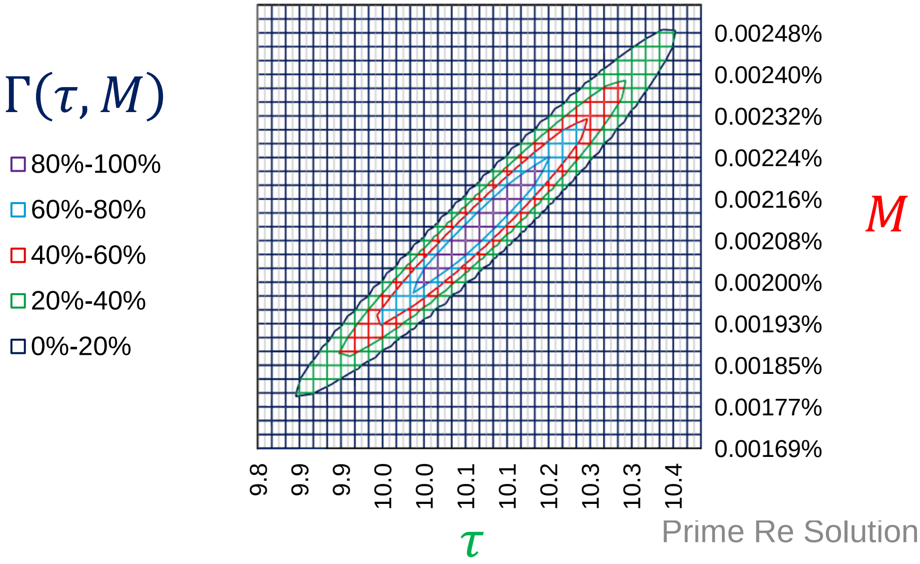
$$\Rightarrow L_x = ce^{-M\tau e^{x/\tau}}$$

$$\Rightarrow q_x = 1 - e^{-M\tau(e^{1/\tau}-1)e^{x/\tau}}$$

- Measurement standard error
  - $\Delta q_x = \sqrt{q_x(1-q_x)/L_x}$

- Chi-square fit
  - $\mathrm{X}^2 = \sum_{x=15}^{100}\left(\frac{q_x - \left(1 - e^{-M\tau(e^{1/\tau}-1)e^{x/\tau}}\right)}{\Delta q_x}\right)^2$





# Mortality models

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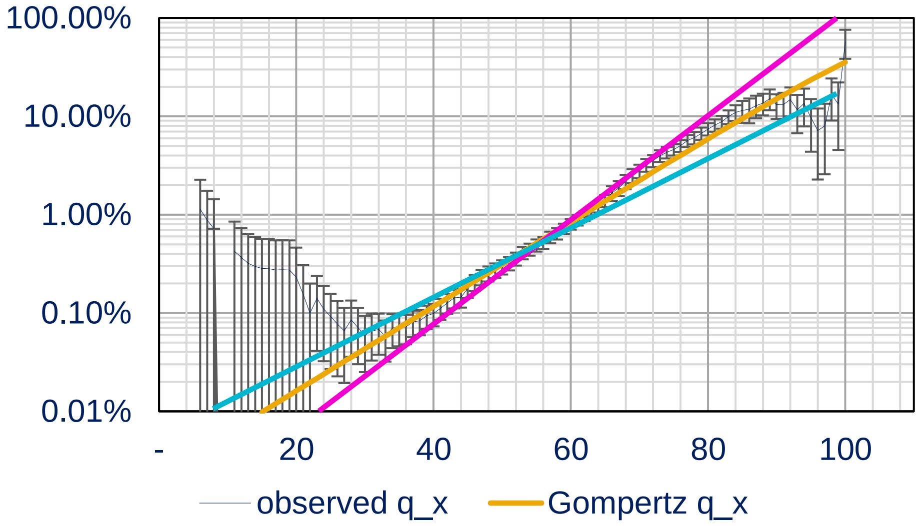
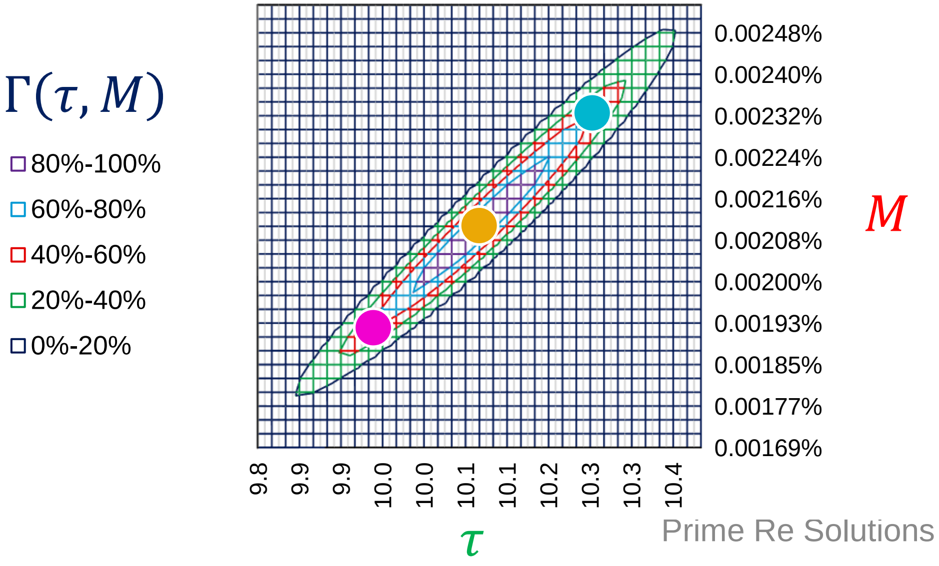




# Agenda

- Parameters

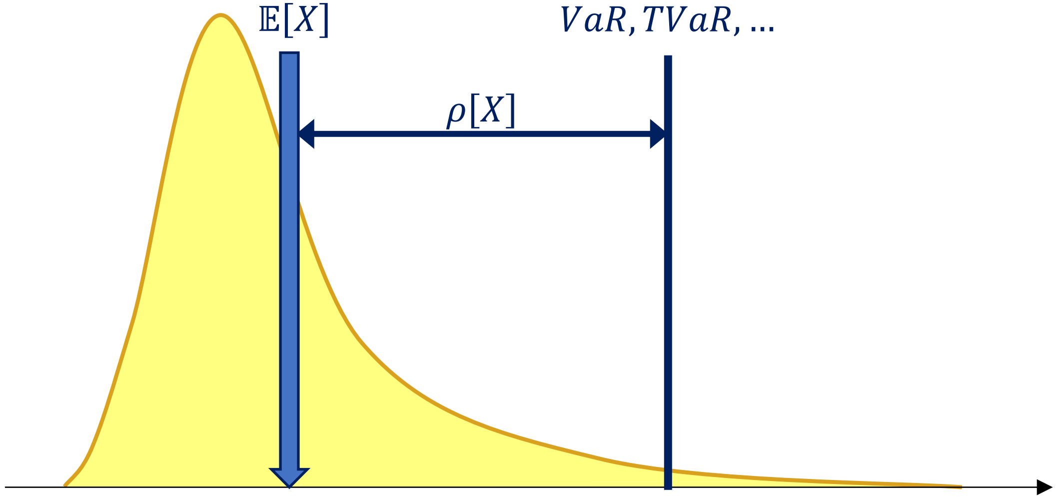
Risk

- Parameter risk

# What is risk?

- The 2 corporate standpoints
  - Accounting view: can only book one number per item
  - Actuarial view: each item is a random variable

- Risk $= \rho[X] =$ deviation from expectation $\mathbb{E}[X]$

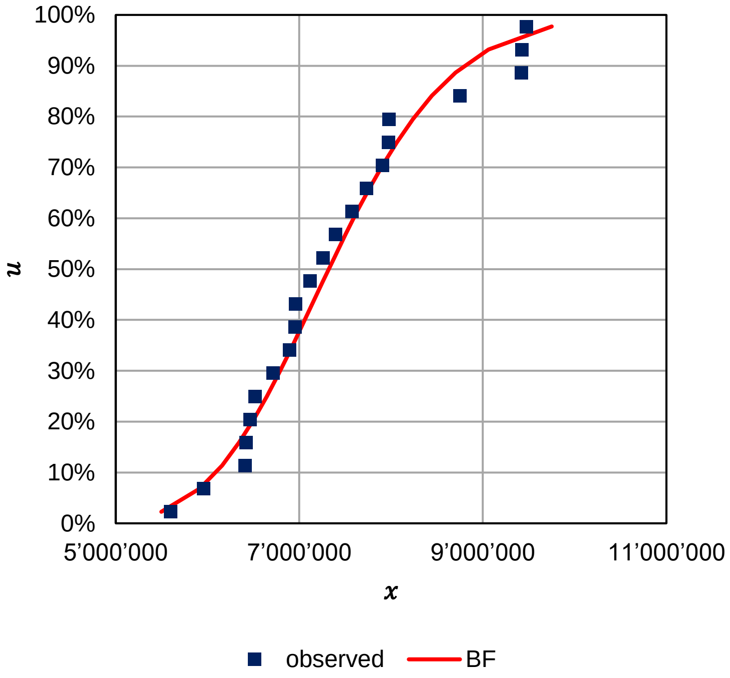


# Risk distribution

- $x_i$ are $n$ actual realizations of a risk $X$ (loss, stock index, …)
    - Empirical CDF is $u_i = u(x_i)$
    - Equiprobable observations $\Rightarrow u_i - u_{i-1} = ctt$

- Assume $X$ distributed according to a probability distribution
    - Its CDF is $F_X(x|\boldsymbol{a}) = \mathbb{P}[X \leq x|\boldsymbol{a}]$
    - It depends on $k$ unknown parameters $\boldsymbol{a}$

- Best fit distribution of risk $X$ $F_X(x|\boldsymbol{a})$



# Agenda

- Parameters
- Risk

Parameter risk

# Risk distribution

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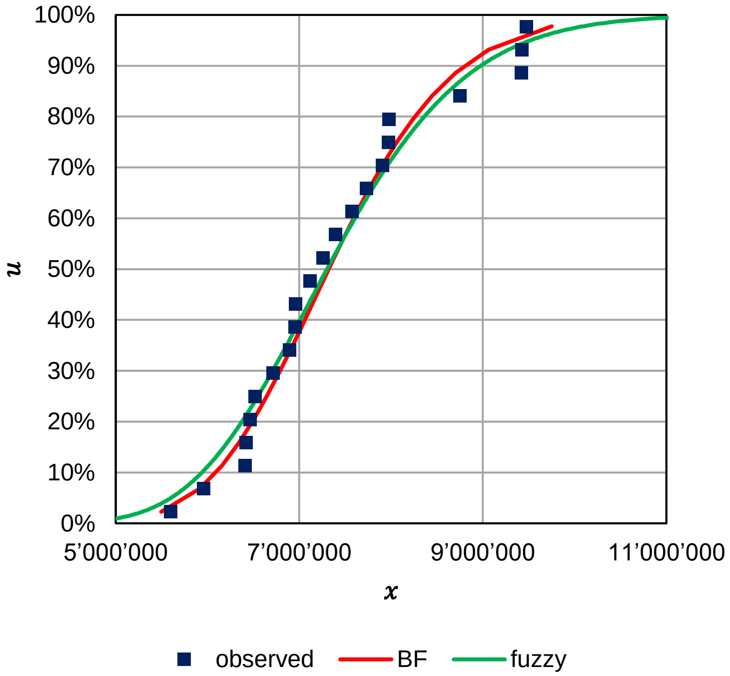
- Assume $X$ distributed according to a probability distribution
  - Its CDF is $F_X(x|\boldsymbol{a}) = \mathbb{P}[X \leq x|\boldsymbol{a}]$
  - It depends on $k$ unknown parameters $\boldsymbol{a}$

- Best fit distribution of risk $X$ $F_X(x|\boldsymbol{a})$
  - Ignores parameter risk

- Fuzzy distribution of risk $X$ $\boxed{F_X(x|\boldsymbol{a}\sim F_{\boldsymbol{a}})}$
  - Accounts for parameter risk
  - Like $NegBin \sim Poisson(\lambda \sim Gamma)$



# Risk distribution – Chi-square fit

- $x_i$ are $n$ actual realizations of a risk $X$ (loss, stock index, …)
  - Empirical CDF is $u_i = u(x_i)$
  - Equiprobable observations ⇒ $u_i - u_{i-1} = ctt$

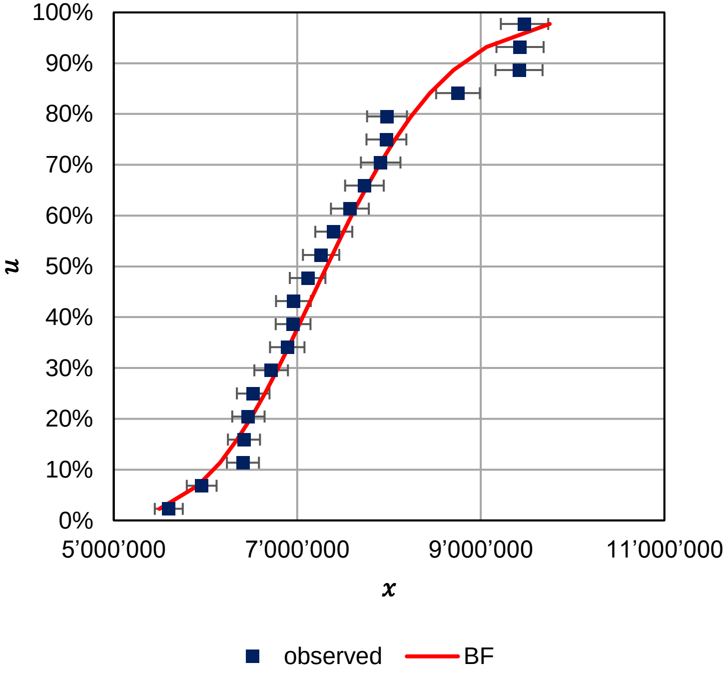
- Assume $X$ distributed according to a probability distribution
  - Its CDF is $F_X(x|\boldsymbol{a}) = \mathbb{P}[X \leq x|\boldsymbol{a}]$
  - It depends on $k$ unknown parameters $\boldsymbol{a}$

- Calibrate the parameters $\boldsymbol{a}$ by minimizing the chi-square function

$$\mathrm{X}^2(\boldsymbol{a}) = \sum_{i=1}^{n} \left( \frac{x_i - \overleftarrow{F}_X(u_i|\boldsymbol{a})}{\Delta x_i} \right)^2$$

$1\sigma$ observation uncertainty

- $\mathrm{X}^2_{min} = \min_{\boldsymbol{a}} \left( \mathrm{X}^2(\boldsymbol{a}) \right)$ yields the best fit parameters $\boldsymbol{a} = (a_1, a_2, \dots, a_k)$



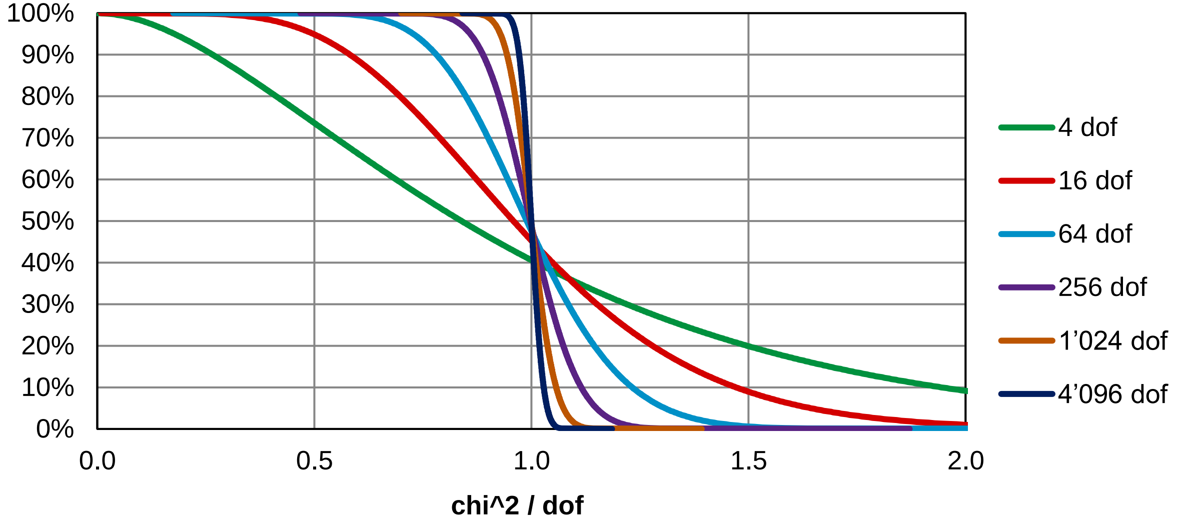
# Risk distribution – goodness of fit

- $\mathrm{X}^2_{min} = \min_{\boldsymbol{a}}\left(\mathrm{X}^2(\boldsymbol{a})\right)$ yields the best fit parameters $\boldsymbol{a} = (a_1, a_2, \dots, a_k)$

- Goodness of fit
  - $\mathrm{X}^2_{min} \sim \chi^2_{n-k}$ chi-square distribution with $n-k$ dof (cf. BIC)
  - $\mathrm{X}^2_{min}$ too large ⇒ probability too small
  - $\frac{\mathrm{X}^2_{min}}{n-k} \gg 1$ ⇒ reject the model
  - $\Delta x_i$ unknown ⇒ scale them s.t. $\frac{\mathrm{X}^2_{min}}{n-k} = 1$

$n =$ number of observations
$k =$ number of parameters

$$\mathrm{X}^2(\boldsymbol{a}) = \sum_{i=1}^{n}\left(\frac{x_i - \overleftarrow{F}_X(u_i|\boldsymbol{a})}{\Delta x_i}\right)^2$$



# Risk distribution – parameters confidence intervals

- $k$ parameters
  - Confidence contour, surface, volume, … at probability $p$: $\mathrm{X}^2\left(\boldsymbol{a}_p\right) = \mathrm{X}^2_{min} + \overleftarrow{\chi}^2_k(p)$
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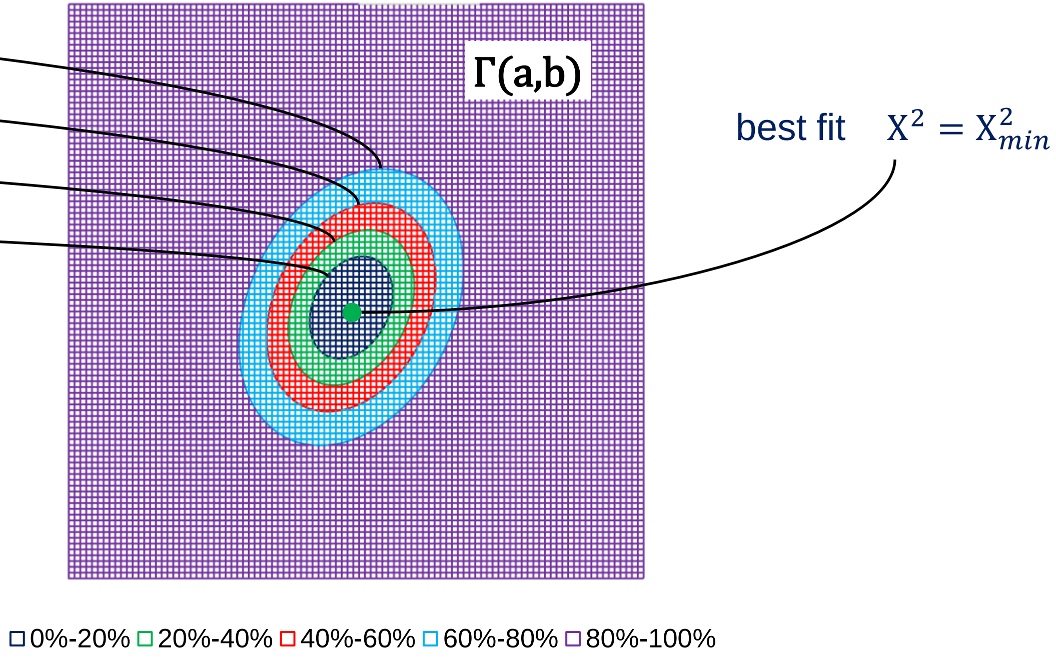
    - …
  - Confidence probability $\Gamma(\boldsymbol{a}) = \chi^2_k\left(\mathrm{X}^2(\boldsymbol{a}) - \mathrm{X}^2_{min}\right)$



# Risk distribution – parameters joint distribution

- $k$ parameters
  - Confidence contour, surface, volume, … at probability $p$: $\mathrm{X}^2(\boldsymbol{a}_p) = \mathrm{X}^2_{min} + \overleftarrow{\chi}^2_k(p)$
    - $\overleftarrow{\chi}^2_2(95\%) = 5.95$
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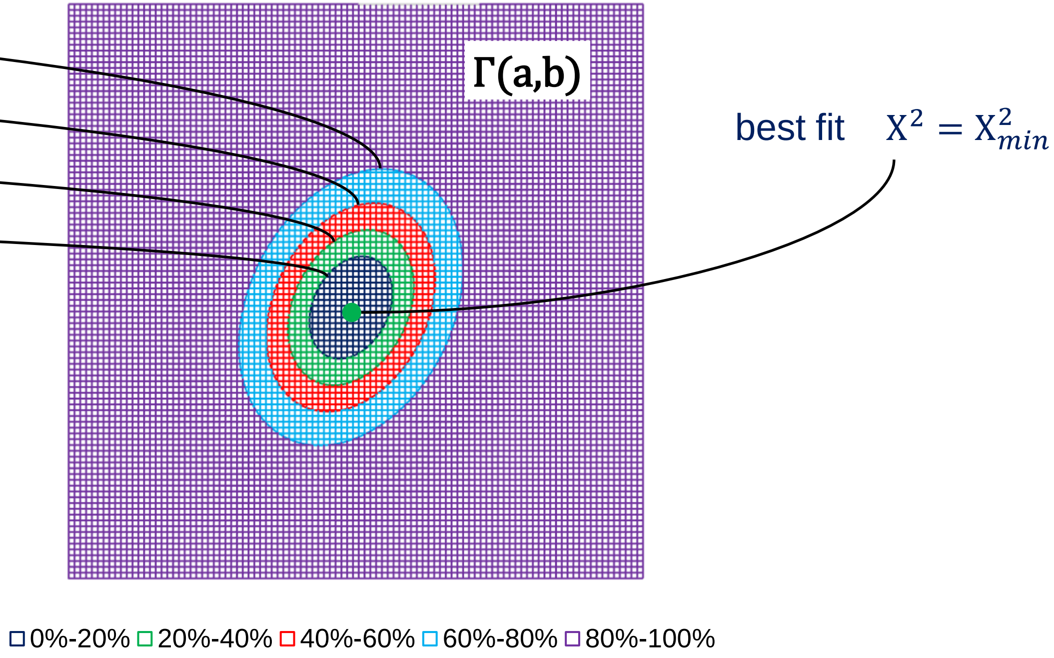
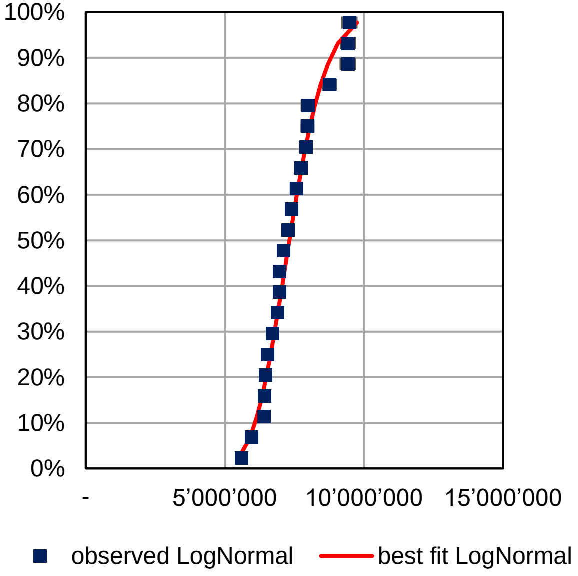
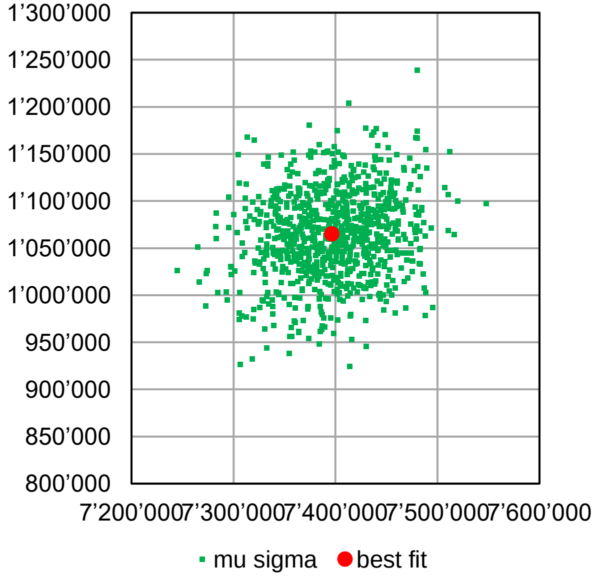
    - $\overleftarrow{\chi}^2_2(20\%) = 0.45$
    - …
  - Confidence probability $\Gamma(\boldsymbol{a}) = \chi^2_k\left(\mathrm{X}^2(\boldsymbol{a}) - \mathrm{X}^2_{min}\right)$
  - Sample numerically the parameters' joint PDF $f(\boldsymbol{a})$
    - e.g. with the brute-force accept-reject algorithm
      - easy 😎
      - inefficient 😪

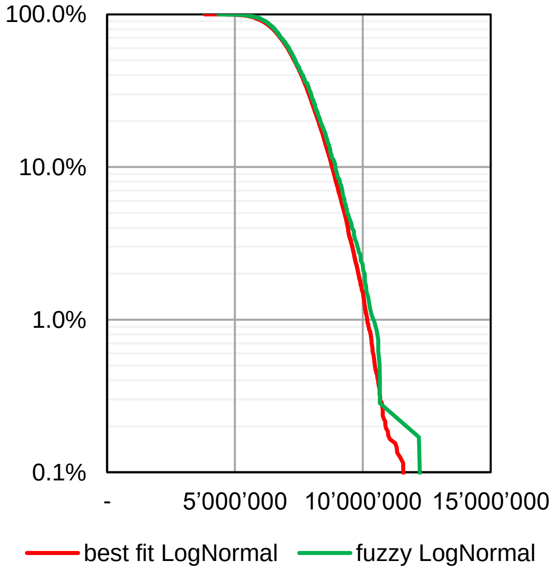


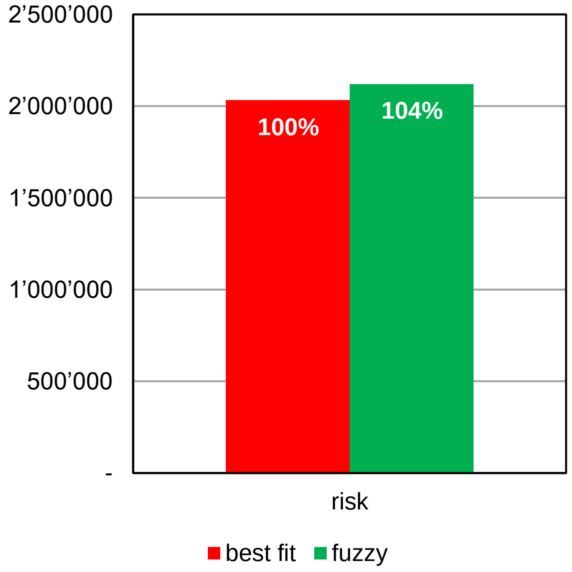
# Modelled risk distribution

- Risk $X \sim$ LogNormal
- Fit LogNormal $F_X$ with $\boldsymbol{a} = (\mu, \sigma)$
- Best fit distribution of risk $X$ $F_X(x|\boldsymbol{a})$
- Fuzzy distribution of risk $X$ $F_X(x|\boldsymbol{a}{\sim}F_{\boldsymbol{a}})$



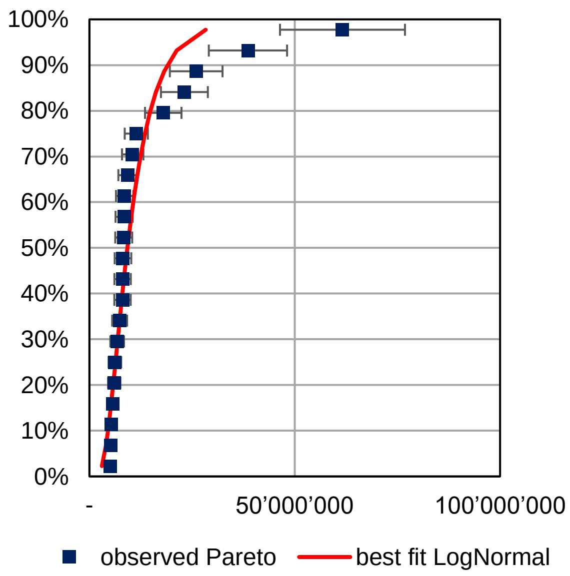


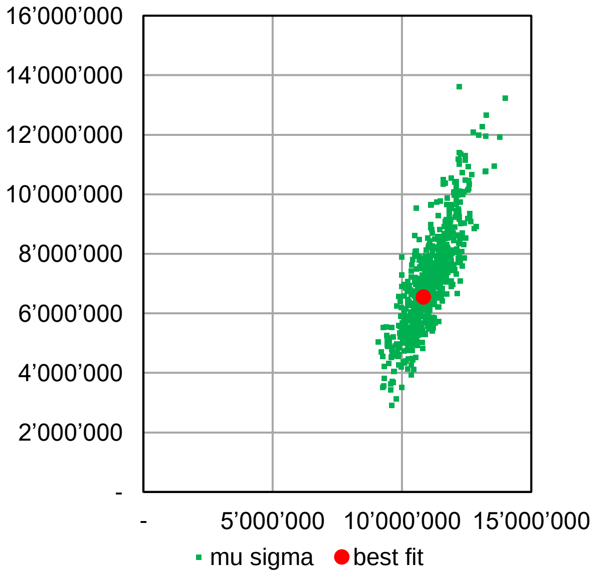


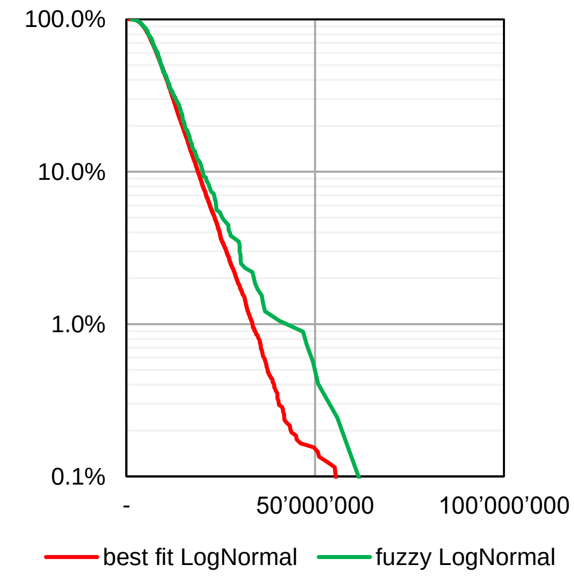


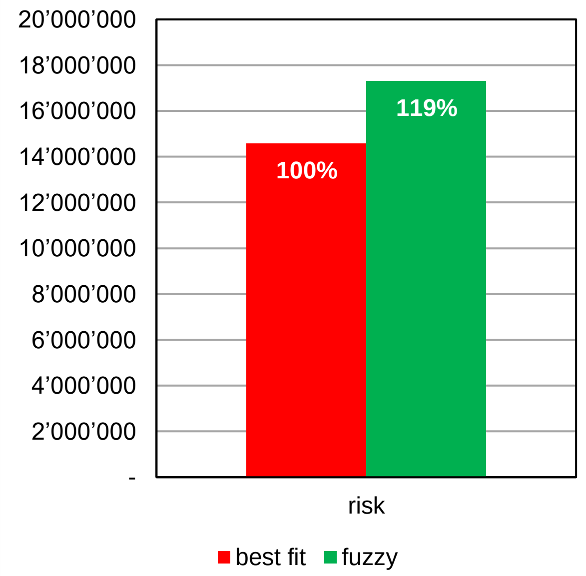
# Modelled risk distribution

- Risk $X \sim$ Pareto
- Fit LogNormal $F_X$ with $\boldsymbol{a} = (\mu, \sigma)$
- Best fit distribution of risk $X$ $F_X(x|\boldsymbol{a})$
- Fuzzy distribution of risk $X$ $F_X(x|\boldsymbol{a} \sim F_{\boldsymbol{a}})$





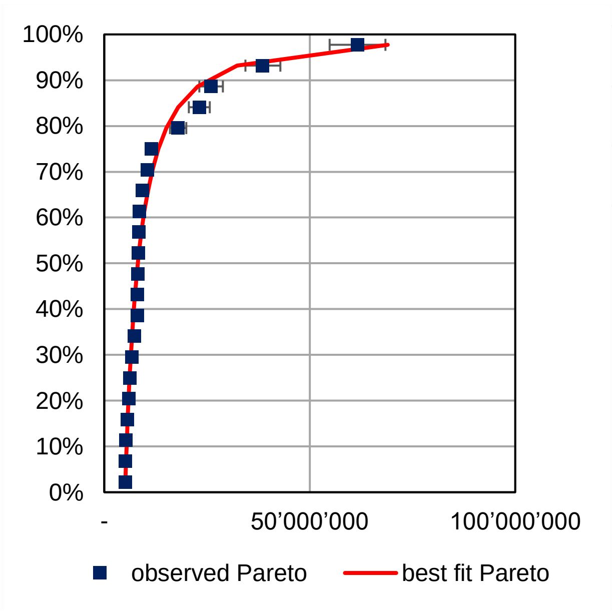


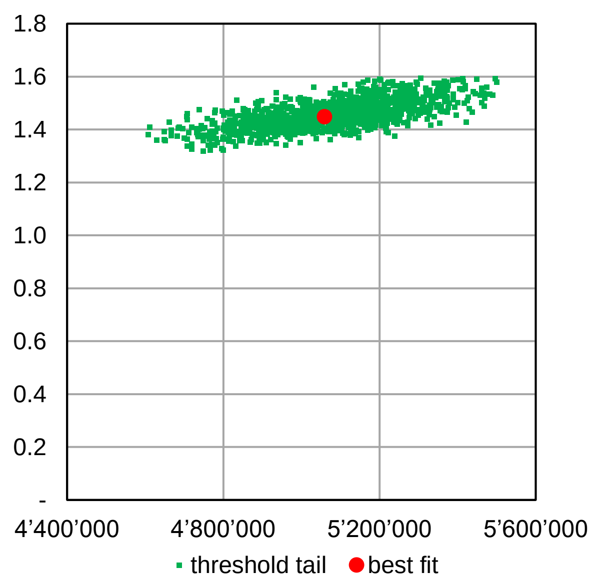


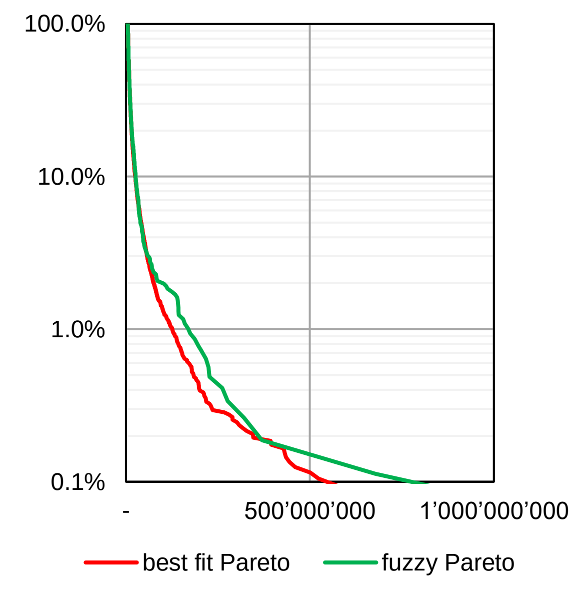
# Modelled risk distribution

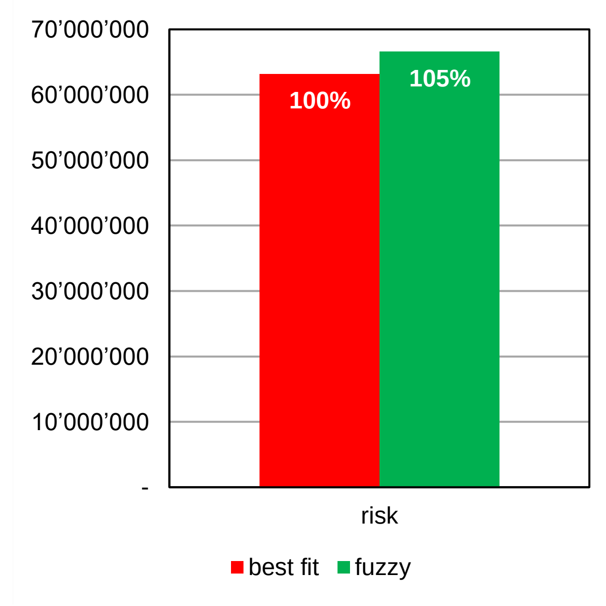
- Risk $X \sim$ Pareto
- Fit Pareto $F_X$ with $\boldsymbol{a} = (T, \alpha)$
- Best fit distribution of risk $X$ $F_X(x|\boldsymbol{a})$
- Fuzzy distribution of risk $X$ $F_X(x|\boldsymbol{a} \sim F_{\boldsymbol{a}})$

$$F_X(x) = 1 - \left(\frac{T}{x}\right)^{\alpha}$$









# Agenda

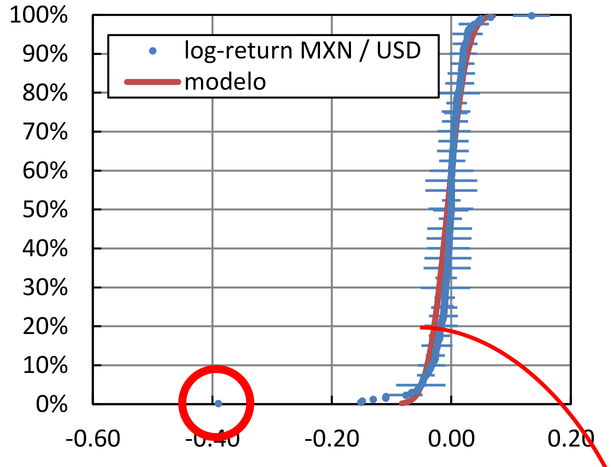
Parameters

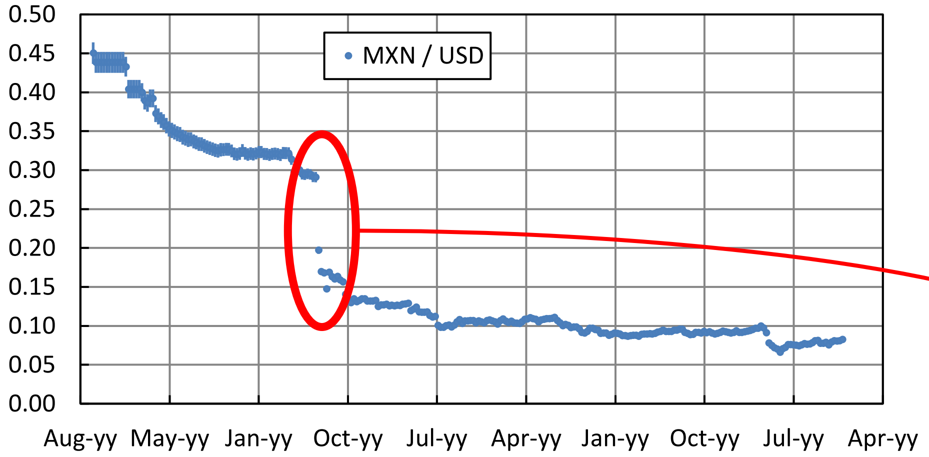
Risk

Parameter risk

# Model risk

the world is not normal 💣

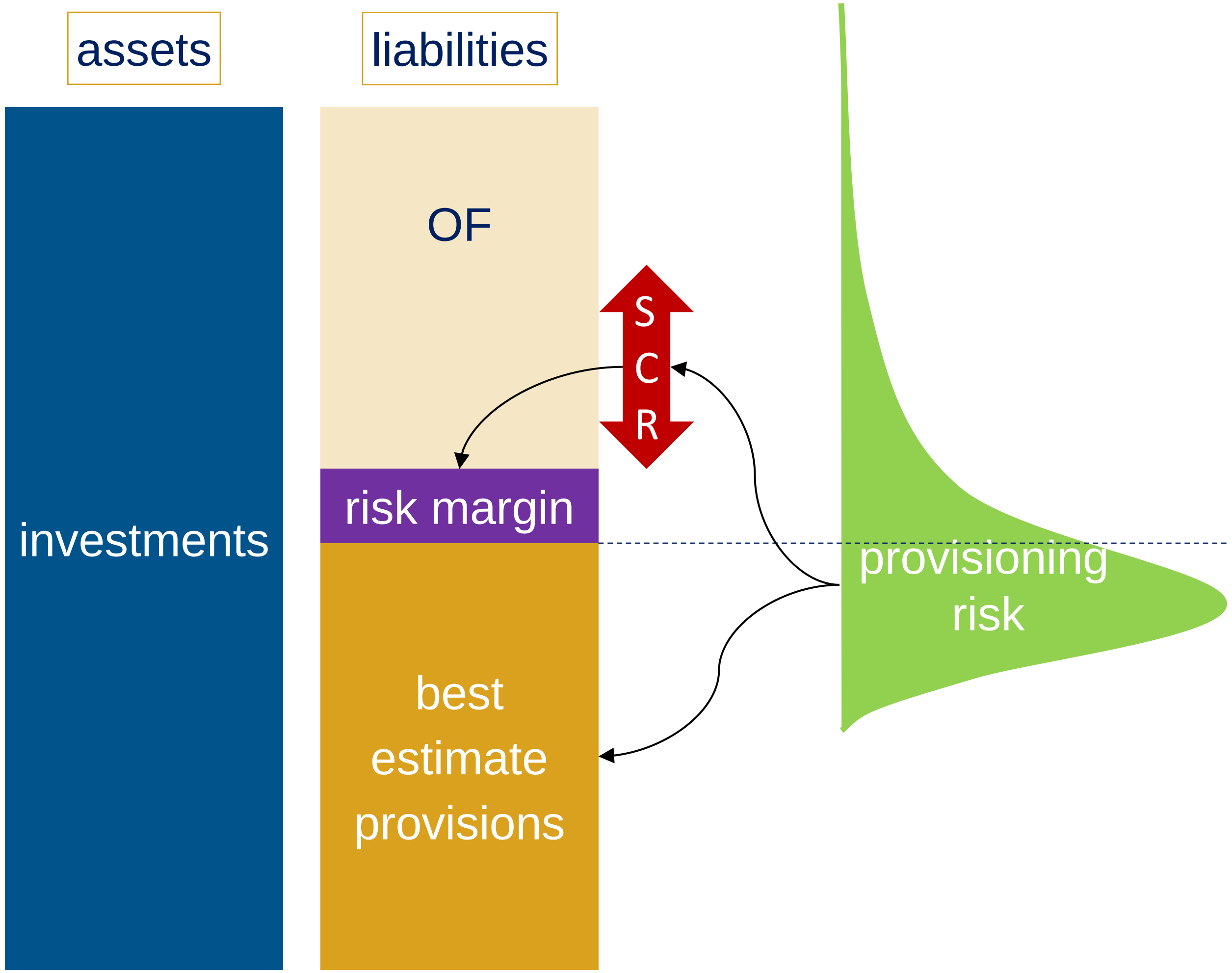




$\chi^2$ / dof = 0.6

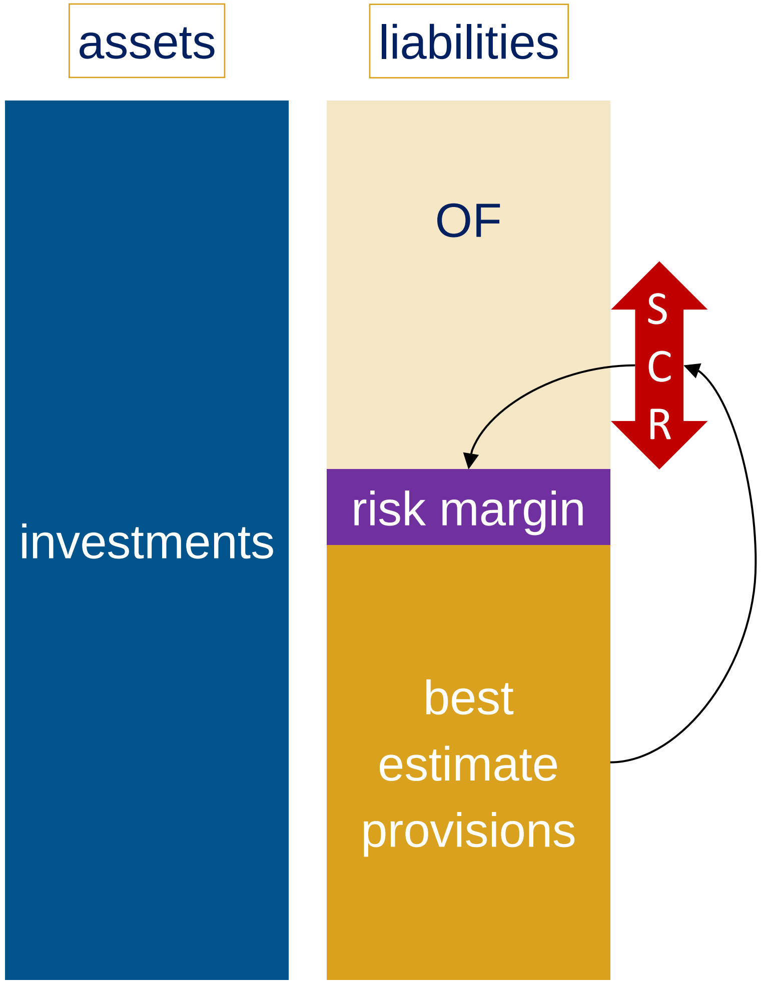
☹

# Et Cartago delenda est…



- Solvency II internal model

# Et Cartago delenda est…



- Solvency II internal model
- Solvency II standard formula

# … et Carthago delenda est!

Solvency I

Solvency II

standard formula

# Contact

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